Classical and Quantum Fermions Linked by an Algebraic Deformation

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Abstract

We study the regular representation ρ_ζ of the single-fermion algebra $\mathcal{A}_\zeta$, i.e., $c^2 = c^{+2} = 0$, $cc^+ + c^+c = \zeta\ 1$, for $\zeta \in [0, 1]$. We show that ρ_0 is a four-dimensional nonunitary representation of $\mathcal{A}_0$ which is faithfully irreducible (it does not admit a proper faithful subrepresentation). Moreover, ρ_0 is the minimal faithfully irreducible representation of $\mathcal{A}_0$ in the sense that every faithful representation of $\mathcal{A}_0$ has a subrepresentation that is equivalent to ρ_0. We therefore identify a classical fermion with ρ_0 and view its quantization as the deformation: $\zeta : 0 \to 1$ of ρ_ζ. The latter has the effect of mapping ρ_0 into the four-dimensional, unitary, (faithfully) reducible representation ρ_1 of $\mathcal{A}_1$ that is precisely the representation associated with a Dirac fermion.

1 Introduction

The description of fermions in terms of the Clifford algebra relations

$$cc^+ + c^+c = 1, \tag{1}$$

$$c^2 = c^{+2} = 0, \tag{2}$$

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dates back to early days of quantum physics. This algebra may be obtained by quantizing a classical system with fermionic variables, e.g., a free fermion or a fermionic oscillator [1, 2]. The classical fermionic variables satisfy the Grassmann algebra relations

$$cc^{+} + c^{+}c = 0, \tag{3}$$

$$c^{2} = c^{+2} = 0. \tag{4}$$

Therefore similarly to the case of bosonic variables, the quantization of a fermionic variable may be viewed as the deformation of the algebraic relations

$$cc^{+} + c^{+}c = \zeta 1, \tag{5}$$

$$c^{2} = c^{+2} = 0, \tag{6}$$

where the deformation parameter ζ takes values in $[0, 1]$. Motivated by the method used in [3] to study the representation theory of orthofermions, we investigate in this paper the effect of the deformation $\zeta \to 0$ on the representations of the associative algebra $\mathcal{A}_\zeta$ generated by 1, c, and c^{+} and subject to relations (5) and (6).

It is well-known [4, 3] that the representations of the Clifford algebra $\mathcal{A}_1$ are, up to equivalence, direct sums of copies of the trivial representation $\rho_{\rm trivial}$:

$$\rho_{\rm trivial}(1) = \rho_{\rm trivial}(c^{+}) = \rho_{\rm trivial}(c) = 0,$$

and the two-dimensional unitary (or $*$-) representation $\rho_\star$:

$$\rho_\star(1) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), \quad \rho_\star(c) = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), \quad \rho_\star(c^{+}) = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right) = \rho_\star(c)^{\dagger}.$$

For $\zeta \neq 0$, one can simply absorb the deformation parameter ζ in the definition of c and/or c^{+}. Therefore the representations of $\mathcal{A}_\zeta$ for $\zeta \neq 0$ are the same as those of $\mathcal{A}_1$. As we shall see below, for $\zeta = 0$ the situation is completely different.

Before, we begin our analysis, we wish to make note of the following facts about the Grassmann algebra $\mathcal{A}_0$.

1. $\mathcal{A}_0$ does not admit nontrivial unitary representations. In order to see this we first note that in view of Eqs. (5) and (6) the algebra $\mathcal{A}_\zeta$ is spanned by the basis elements

$1, c^+, c$ and n, where $n := c^+c$. Now, let $(\mathcal{H}, \langle \ , \ \rangle)$ be an inner-product space and $\rho : \mathcal{A}_0 \to \mathrm{End}(\mathcal{H})$ be a representation of $\mathcal{A}_0$ where 'End' abbreviates 'Endomorphism' (a linear operator mapping $\mathcal{H}$ into $\mathcal{H}$.) By definition, if ρ is a unitary representation, then $\rho(c^+) = \rho(c)^\dagger$, where a dagger stands for the adjoint of the corresponding operator. According to Eqs. (3) and (4), the unitarity of ρ implies for all $|\psi\rangle \in \mathcal{H}$,

$$||\rho(n)|\psi\rangle||^2 = \langle\psi|\rho(n)^\dagger\rho(n)|\psi\rangle = \langle\psi|\rho(n)^2|\psi\rangle = \langle\psi|\rho(n^2)|\psi\rangle = 0.$$

Hence $\rho(n)|\psi\rangle = 0$. On the other hand,

$$||\rho(c)|\psi\rangle||^2 = \langle\psi|\rho(c)^\dagger\rho(c)|\psi\rangle = \langle\psi|\rho(n)|\psi\rangle = 0.$$

Therefore for all $|\psi\rangle \in \mathcal{H}$, $\rho(c)|\psi\rangle = 0$, so that $\rho(c) = 0$, $\rho(c^+) = 0$, and ρ is trivial.

2. The only irreducible representation of $\mathcal{A}_0$ is the one-dimensional representation defined by

$$\rho^{(1)}_\emptyset(1) = 1, \qquad \rho^{(1)}_\emptyset(c^+) = \rho^{(1)}_\emptyset(c) = 0. \tag{7}$$

To see this let $\rho : \mathcal{A}_0 \to \mathrm{End}(V)$ be an arbitrary representation. Then $V_\emptyset = \mathrm{Im}(\rho(n)) := \{\rho(n)v|v \in V\}$ is an invariant (ρ-stable) subspace [5], because for all $x \in \mathcal{A}_0$ and for all $v \in V_\emptyset$, $\rho(x)v \in V_\emptyset$. This shows that ρ is reducible. Furthermore, the subrepresentation obtained by restricting ρ to $V_\emptyset$ is clearly equivalent to $\rho^{(1)}_\emptyset$.

Next, consider the regular representation $\rho_\zeta : \mathcal{A}_\zeta \to \mathrm{End}(\mathcal{A}_\zeta)$ of $\mathcal{A}_\zeta$ that is defined by

$$\forall x, y \in \mathcal{A}_\zeta, \qquad \rho_\zeta(x)y := xy. \tag{8}$$

In the basis $\{1, c^+, c, n\}$, where

$$1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c^+ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad n = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

we have

$$\rho_\zeta(1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_\zeta(c^+) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tag{9}$$

$$\rho_\zeta(c) = \begin{pmatrix} 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \zeta \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \rho_\zeta(n) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \zeta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \zeta \end{pmatrix}. \tag{10}$$

Here we have made use of Eqs. (5), (6), and (8).

It is not difficult to show that $\zeta \neq 0$ if and only if ρ_ζ is a pseudo-unitary representation [6]. This is equivalent to the requirement that there is a linear Hermitian invertible operator η such that

$$\rho_\zeta(c^+) = \rho_\zeta(c)^\sharp := \eta^{-1}\rho_\zeta(c)^\dagger\eta. \tag{11}$$

This can be easily checked by taking η to be an arbitrary 4×4 matrix and imposing the condition $\eta\rho_\zeta(c^+) = \rho_\zeta(c)^\dagger\eta$ to determine the matrix elements of η. It follows that the determinant of η is proportional to ζ. Therefore ρ_0 is not pseudo-unitary. For $\zeta \neq 0$ there are many invertible matrices η satisfying (11), e.g.,

$$\eta = \begin{pmatrix} 0 & \zeta^{-1} & \zeta^{-1} & 0 \\ \zeta^{-1} & 0 & 0 & 1 \\ \zeta^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{12}$$

Furthermore, in this case, there are similarity transformations

$$\rho_\zeta(x) \to \rho'_\zeta(x) := S^{-1}\rho_\zeta(x)S \tag{13}$$

that reduce ρ_ζ into the direct sum of two nontrivial two-dimensional irreducible represen-

tations. A convenient choice is

$$S = \begin{pmatrix} \zeta & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{14}$$

Using Eqs. (9), (10), (13), and (14), we have

$$\rho'_\zeta(1) = \begin{pmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 1 & 0 \\ & & 0 & 1 \end{pmatrix}, \qquad \rho'_\zeta(c^+) = \begin{pmatrix} 0 & 0 & & \\ \zeta & 0 & & \\ & & 0 & 0 \\ & & 1 & 0 \end{pmatrix}, \tag{15}$$

$$\rho'_\zeta(c) = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & & \\ & & 0 & \zeta \\ & & 0 & 0 \end{pmatrix}, \qquad \rho'_\zeta(n) = \begin{pmatrix} 0 & 0 & & \\ 0 & \zeta & & \\ & & 0 & 0 \\ & & 0 & \zeta \end{pmatrix}, \tag{16}$$

where the empty entries are zero. Clearly ρ'_1 is the direct product of two copies of the basic unitary representation $\rho_\star$ of the Clifford algebra $\mathcal{A}_1$. Also note that for $\zeta = 0$ the matrix S is not invertible, and the above construction does not apply.

In fact, it is not difficult to show that the Grassmann algebra $\mathcal{A}_0$ does not admit one, two, or three-dimensional representations that are faithful. In order to see this, consider an arbitrary representation $\rho : \mathcal{A}_0 \to \mathrm{End}(V)$ where V is a complex (or real) vector space, and suppose that ρ is faithful (one-to-one). Then there is $v_1 \in V$ such that $v_4 := \rho(n)v_1 \neq 0$. This together with the fact that $\rho(n) = \rho(c^+)\rho(c)$ imply $v_2 := \rho(c^+)v_1 \neq 0$ and $v_3 := \rho(c^+)v_1 \neq 0$. Next let $\lambda_i \in \mathbb{C}$, with $i \in \{1, 2, 3, 4\}$, satisfy

$$\sum_{i=1}^{4} \lambda_i \, v_i = 0. \tag{17}$$

Applying $\rho(n)$ to both sides of this equation yields $\lambda_1 = 0$. Substituting this equation in (17) and acting by $\rho(c)$ and $\rho(c^+)$ on both sides of the resulting equation lead to $\lambda_2 = 0$ and $\lambda_3 = 0$, respectively. Therefore $\lambda_i = 0$ for all $i \in \{1, 2, 3, 4\}$; v_i are linearly independent,

and $\dim(V) \geq 4$. This in particular shows that the regular representation ρ_0 is the 'lowest' dimensional faithful representation. In the following we shall use the term '*faithfully irreducible representation*' by which we mean a faithful representation that does not admit a proper faithful subrepresentaion. Note that a faithfully irreducible representation may very well be reducible. The typical example is the regular representation ρ_0.

Next, consider the span of v_i:

$$V_{v_1} := \mathrm{Span}(v_1, v_2, v_3, v_4) = \left\{ \sum_{i=1}^{4} \lambda_i \, v_i \;\middle|\; \lambda_i \in \mathbb{C} \right\}.$$

It is not difficult to see that for all $v \in V_{v_1}$ and $x \in \mathcal{A}_0$, $\rho(x)v \in V_{v_1}$. Hence the restriction $\rho_{v_1} : \mathcal{A}_0 \to \mathrm{End}(V_{v_1})$ of ρ to V_{v_1}, which is defined by

$$\forall x \in \mathcal{A}_0 \;\; \text{and} \;\; \forall v \in V_{v_1}, \qquad \rho_{v_1}(x)v := \rho(x)v,$$

provides a representation of $\mathcal{A}_0$. Clearly, ρ_{v_1} is equivalent to the regular representation ρ_0. This proves the following.

Theorem: *Every faithful representation of the Grassmann algebra $\mathcal{A}_0$ has a subrepresentation that is equivalent to the regular representation ρ_0. In particular, ρ_0 is (up to equivalence) the unique 4-dimensional faithfully irreducible representation of $\mathcal{A}_0$.*

This is analogous to the well-known fact about the Clifford algebra $\mathcal{A}_1$, namely that every faithful representation of $\mathcal{A}_1$ has a subrepresentation that is equivalent to the canonical representation $\rho_\star$. In particular, $\rho_\star$ is (up to equivalence) the unique 2-dimensional faithful irreducible representation of $\mathcal{A}_1$. However there is a stronger result [3] indicating that every representation of $\mathcal{A}_1$ is a direct product of copies of the trivial representation ρ_{trivial} and the canonical representation $\rho_\star$. A similar result does not hold for $\mathcal{A}_0$. This is mainly because there are, besides the trivial representation, one, two and three-dimensional non-faithful representations, namely $\rho_\emptyset^{(1)} : \mathcal{A}_0 \to \mathrm{End}(\mathbb{C}) = \mathbb{C}$ of (7) and $\rho_\emptyset^{(2)} : \mathcal{A}_0 \to \mathrm{End}(\mathbb{C}^2)$ and $\rho_\emptyset^{(3)} : \mathcal{A}_0 \to \mathrm{End}(\mathbb{C}^3)$ defined by

$$\rho_\emptyset^{(2)}(1) = 1, \qquad \rho_\emptyset^{(2)}(c) = 0, \qquad \rho_\emptyset^{(2)}(c^+) = \mu, \tag{18}$$

$$\rho_\emptyset^{(3)}(1) = 1, \qquad \rho_\emptyset^{(3)}(c) = \nu, \qquad \rho_\emptyset^{(3)}(c^+) = \nu^+, \tag{19}$$

where

$$\mu := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \nu := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \nu^+ := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

In view of the above-stated uniqueness property of the regular representation ρ_0 of the Grassmann algebra $\mathcal{A}_0$, we propose to identify a 'classical fermion' with ρ_0. Then the quantization of ρ_0 may be viewed as the deformation $\zeta : 0 \to 1$ of the regular representation ρ_ζ of the one-fermion algebra $\mathcal{A}_\zeta$ that maps the classical fermion ρ_0 to the 'quantum fermion' ρ_1. The latter is a four-dimensional unitary reducible representation of the Clifford algebra $\mathcal{A}_1$ that is associated with a Dirac fermion. In this sense Dirac fermions are naturally linked with the quantization of the classical fermions.

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References

[1] F. A. Berezin and M. S. Marinov, Ann. Phys. (N. Y.) **104**, 336 (1977).

[2] B. DeWitt, *Supermanifolds* (Cambridge University Press, Cambridge, 1992).

[3] A. Mostafazadeh, J. Phys. A: Math. Gen. **34**, 8601 (2001).

[4] H. B. Lawson, Jr. and M.-L. Michelsohn, *Spin geometry* (Princeton University Press, Princeton, 1989).

[5] J. M. G. Fell and R. S. Doran, *Representations of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles* (Academic Press, San Diego, 1988).

[6] A. Mostafazadeh, preprint: math-ph/0302050, to appear in J. Math. Phys.; See also A. Mostafazadeh, J. Math. Phys. **43**, 205 (2002).